FINITELY ADDITIVE MEASURES IN THE ERGODIC THEORY OF MARKOV CHAINS. II †

A. I. Zhdanok*

Abstract

We develop a new approach to the study of general Markov chains (MC), i.e., homogeneous Markov processes with discrete time on an arbitrary phase space. In the first part of the article, we suggested an extension of the traditional space of countably additive measures to the space of finitely additive measures. Given an arbitrary phase space, we constructed its "gamma-compactification" to which we extended each Markov chain. We established an isomorphism between all finitely additive Markov chains on the initial space and Feller countably additive chains on its "gamma-compactification." Using the above construction, in the second part, we prove weak and strong ergodic theorems that establish a substantial dependence of the asymptotic behavior of a Markov chain on the presence and properties of invariant finitely additive measures. The study in the article is carried out in the framework of functional operator approach.

Key words and phrases: finitely additive measure, countably additive measure, Markov chain, Markov operators, arbitrary phase space, compactification of an arbitrary phase space, extension of a Markov chain to the compactification, invariant measure, ergodic theorems.

The present article is a continuation of [21]; hence, all notations remain the same and the definitions are repeated only in a few cases. All references to sections and formulas of the first part [21] are straightforward.

4. ERGODIC THEOREMS FOR MARKOV CHAINS

Introduction

Let X be an arbitrary set and let Σ be an algebra of its subsets. Denote by $\sigma(\Sigma)$ the σ-algebra generated by Σ, often assuming Σ itself to be a σ-algebra. If X is a topological space with topology $\tau = \tau_X$ then $\mathcal{A} = \mathcal{A}_X = \mathcal{A}_\tau$

*Tyva State University, Kyzyl, Republic of Tyva, RUSSIA. E-mail address: Zhdanok@tuva.ru

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and $\mathcal{B} = \mathcal{B}_X = \mathcal{B}_\tau$ are the Borel algebra and σ-algebra on X generated by τ. Throughout the article, we assume Σ to contain all singletons of X. We also assume that our topological space X is minimally T_1-separated, i.e., we suppose that all its singletons are closed. In this case, its Borel algebra $\mathcal{A}$ and σ-algebra $\mathcal{B}$ contain all singletons. All Hausdorff, regular, normal, and metric spaces are T_1-separated.

We recall some notations of [21]. For an arbitrary X, denote by $B(X)$ the Banach space of all bounded functions $f\colon X \to \mathbb{R}$ with the sup-norm; by $H(X,\Sigma)$, the linear space of all linear combinations of the characteristic functions χ_E of $E \in \Sigma$; by $B(X,\Sigma)$, the closure of $H(X,\Sigma)$ in $B(X)$. If Σ is a σ-algebra then $B(X,\Sigma)$ is the Banach space of all bounded Σ-measurable functions $f\colon X \to \mathbb{R}$ with the sup-norm.

In this article, we also use the following notations: $C(X)$ is the Banach space of all continuous bounded functions on a topological space X with the sup-norm; $ba(X,\Sigma)$ is the Banach space of all bounded finitely additive measures $\mu\colon \Sigma \to \mathbb{R}$ with norm the total variation of a measure on X $\big(\|\mu\| = \mathrm{Var}(\mu, X)\big)$; $ca(X,\Sigma)$ is the subspace of all countably additive measures; $pfa(X,\Sigma)$ is the subspace of all purely finitely additive measures; $rba(X,\Sigma)$ and $rca(X,\Sigma)$ are the subspaces of all regular measures for a topological space X.

Suppose that we have a countably additive Markov chain (MC) on (X,Σ) with transition function $p(x,E)$ and Markov operators $T\colon B(X,\Sigma) \to B(X,\Sigma)$ and $A\colon ca(X,\Sigma) \to ca(X,\Sigma)$. In [21, Section 3], we constructed an extension of A from the "traditional" domain of definition $ca(X,\Sigma)$ to the space of finitely additive measures $ba(X,\Sigma)$. Moreover, the extended operator A is adjoint to T, i.e., $T^* = A$.

If M is a space of measures then

$$S_M = \big\{\mu \in M : \mu \geq 0,\ \mu(X) = 1\big\},$$
$$\Delta_M = \{\mu \in S_M : \mu = A\mu\}.$$

In particular,

$$\Delta_{pfa} = \big\{\mu \in S_{ba} : \mu = A\mu,\ \mu \text{ is purely finitely additive}\big\}.$$

We will sometimes omit the index for Δ_{ba}: $\Delta = \Delta_{ba}$.

The spaces of functions and measures are in a sense dual: $B^*(X,\Sigma) = ba(X,\Sigma)$ for an arbitrary (X,Σ) and $C^*(X) = rba(X,\mathcal{A})$ for a normal topological X, $C^*(X) = rca(X,\mathcal{B})$ for a compact Hausdorff X, where equalities stand for isometric isomorphisms and the spaces on the left are the topological duals to the corresponding spaces of functions.

By the gamma-compactification (see [21, Section 9]) $\gamma X = \gamma_\Sigma X$ of the initial measure space (X, Σ) we mean the space of all maximal ideals of the Banach algebra $B(X, \Sigma)$ with the Tychonoff topology τ_γ and the Borel σ-algebra $\mathcal{B}_\gamma = \sigma(\tau_\gamma)$. Also, the gamma-compactification can be treated as the family of all multiplicative functionals in $B^*(X, \Sigma) = ba(X, \Sigma)$, to which there correspond two-valued measures in the $*$-weak topology τ_B.

Consider the initial space (X, Σ) and its gamma-compactification $(\gamma X, \mathcal{B}_\gamma)$. The mapping $s\colon X \to \gamma X$ is an injective dense embedding of X into γX (by identifying x and $s(x)$). The mapping $t\colon \Sigma \to \mathcal{N}_{\gamma X}$ is an algebraic isomorphism of the σ-algebra Σ onto the algebra $\mathcal{N}_{\gamma X}$ of clopen sets in γX. Moreover, $\mathcal{N}_{\gamma X}$ is a base of topology on $(\gamma X, \tau_\gamma)$ and $\sigma(\mathcal{N}_{\gamma X}) = \mathcal{B}_Z(\gamma X)$, where $\mathcal{B}_Z(\gamma X)$ is the Baire σ-algebra of γX (see [21, Definition 9.2]).

There is an isometric linear isomorphism $r\colon B(X, \Sigma) \to C(\gamma X)$ that realizes a continuous extension of measurable functions on (X, Σ) to continuous functions on $(\gamma X, \tau_\gamma)$. Adjoint to r is the isometric linear isomorphism $r^*\colon rca(\gamma X, \mathcal{B}_\gamma) \to ba(X, \Sigma)$. It is often useful to consider the inverse isomorphism $[r^*]^{-1}\colon ba(X, \Sigma) \to rca(\gamma X, \mathcal{B}_\gamma)$ extending finitely additive measures on (X, Σ) to countably additive regular measures on $(\gamma X, \mathcal{B}_\gamma)$.

The ergodic theory for Feller Markov chains (MC's) on a compact space is well developed in contrast to the case for arbitrary chains on an arbitrary phase space. In Section 10, we constructed an isomorphism between an arbitrary MC and a Feller MC on a compact space. In the next sections, this construction will enable us to translate many facts in the theory of Feller MC's into the corresponding assertions for arbitrary MC's. As a result, we will obtain new ergodic theorems with purely finitely additive measures actively involved. Special techniques developed in the first part of the article allow us to make the ergodic theorems themselves and their proofs clearer and formally shorter.

There are many articles considering the construction of the Feller extension of the initial Markov chain to the Stone–Čech or a similar compactification of the initial phase space. Here we must point out a series of articles by Foguel, for example, [5–7], and also the articles [9] and [14] by Le Cam and Shur. In Horowitz's article [8], an extension of the MC is in fact constructed to some compactification generated by the space $L_\infty(X, \Sigma, m)$ with a prescribed measure m. A. A. Borovkov also uses extensions in proving the ergodic theorems in his monograph [1]. There are some more publications on the topic, in which special constructions are considered of extensions of MC's to enlargements of the phase space.

We stress that, in contrast to all similar articles, our approach imposes a priori restrictions neither on the Markov chain nor on its phase space. We will make comparison with other authors' results directly in the relevant places.

11. Ergodic Alternatives

Recall that we denote by λ_n^μ the Cesàro mean for a Markov sequence of measures with initial measure $\mu \in ba(X, \Sigma)$:

$$\lambda_n^\mu = \lambda_n = \frac{1}{n} \sum_{k=1}^{n} A^k \mu, \quad n = 1, 2, \dots .$$

We now use conventional notation of the literature on probability:

$$p^1(x, E) = p(x, E),$$

$$p^{k+1}(x, E) = \int_X p^k(y, E) p(x, dy) = \langle p^k(\cdot, E), p(x, \cdot) \rangle, \quad k = 1, 2, \dots .$$

Here the upper index in $p^k(x, E)$ means the integral "convolution." Then $A^k \mu = \int_X p^k(x, \cdot) \mu(dx)$.

Theorem 11.1. *Assume given an* MC *on* (X, Σ). *For every set* $E \in \Sigma$, *we have the equalities* $r_n^1(E) = r_n^2(E) = r_n^3(E) = r_n^4(E)$, $n = 1, 2, \dots,$ *where*

$$r_n^1(E) = \sup_{\mu \in S_{ba}} \lambda_n^\mu(E); \qquad r_n^2(E) = \sup_{\mu \in S_{ca}} \lambda_n^\mu(E);$$

$$r_n^3(E) = \sup_{x \in X} \frac{1}{n} \sum_{k=1}^{n} p^k(x, E); \qquad r_n^4(E) = \left\| \frac{1}{n} \sum_{k=1}^{n} T^k \chi_E \right\|_{B(X,\Sigma)}.$$

Proof. Let $\mu \in S_{ba}$. Then

$$\lambda_n^\mu(E) = \frac{1}{n} \sum_{k=1}^{n} \int p^k(x, E) \mu(dx) = \int \left[\frac{1}{n} \sum_{k=1}^{n} p^k(x, E) \mu(dx) \right]$$
$$\le \sup_{x \in X} \frac{1}{n} \sum_{k=1}^{n} p^k(x, E) = r_n^3(E) = \sup_{x \in X} \frac{1}{n} \sum_{k=1}^{n} A^k \delta_x(E)$$
$$\le \sup_{\lambda \in S_{ba}} \frac{1}{n} \sum_{k=1}^{n} A^k \lambda(E) = r_n^1(E)$$

(here and below δ_x is the Dirac measure with support at x). From this it follows that

$$r_n^1(E) = \sup_{\mu \in S_{ba}} \lambda_n^\mu(E) \le r_n^3(E) \le r_n^1(E),$$

i.e., $r_n^1(E) = r_n^3(E)$.

Reasoning in a similar way, we find $r_n^2(E) = r_n^3(E)$. The equality $r_n^3(E) = r_n^4(E)$ follows from definitions. The theorem is proven.

Theorem 11.2. *Suppose that* X *is a topological space, and we have an arbitrary* MC *on* $(X, \mathcal{B})$. *Put*

$$r_n^5(E) = \sup_{\mu \in S_{rca}} \lambda_n^\mu(E), \quad E \in \mathcal{B}, \; n = 1, 2, \dots .$$

Then $r_n^i(E) = r_n^5(E)$, $i = 1, 2, 3, 4$, *for every* $E \in \mathcal{B}$.

The proof is carried out by analogy with that of the preceding theorem.

Remark. If X is a topological space then we may assume a Markov chain to be defined on the pair $(X, \mathcal{A})$ with $\mathcal{A}$ the Borel algebra. In this case, we put

$$r_n^6(E) = \sup_{\mu \in S_{rba}} \lambda_n^\mu(E), \quad E \in \mathcal{A}, \; n = 1, 2, \dots .$$

It is easy to check that $r_n^i(E) = r_n^6(E)$, $i = 1, 2, 3, 4, 5$, for every $E \in \mathcal{A}$.

Theorem 11.3. *Let* X *be a normal space. Assume that we have a Feller* MC *on* $(X, \mathcal{B})$. *Then each closed set* $F \subset X$ *always meets either* (A1) *or* (B1):

(A1) $\forall \mu \in S_{rba} \;\; \lambda_n^\mu(F) \to 0$ *as* $n \to \infty$;

(B1) $\exists \mu \in S_{rba} \;\; \mu = A\mu$, $\mu(F) > 0$.

Proof. Assume that (A1) is not fulfilled for F. Then there exist $\mu \in S_{rba}$, $\delta > 0$, and a sequence $\{n_i\}$ such that $\lambda_{n_i}^\mu(F) \geq \delta$, $i = 1, 2, \dots$. By Theorem 7.1 (Corollary 7.2), $\{\lambda_{n_i}^\mu\}$ has τ_C-limit measures η all invariant for A. Theorem 2.9 implies that $\eta(F) \geq \delta > 0$, i.e., (B1) holds. If, in turn, (B1) holds then, obviously, (A1) does not. The theorem is proven.

Corollary 11.1. *Under the conditions of Theorem* 11.3, *let* $F \in \mathcal{A}$ *be an arbitrary set. If* (A1) *does not hold for* F *then its closure* $\overline{F}$ *satisfies* (B1). *Moreover, in this case,* F *itself need not meet* (B1).

Theorem 11.4. *Let* X *be a normal space. Assume given a Feller* MC *on* $(X, \mathcal{B})$. *Then, for every compact set* $F \subset X$, (A1) *is equivalent to* (A2) *and* F *always meets either* (A1), (A2) *or* (B2), *where*

(A2) $r_n^i(F) \to 0$ *as* $n \to \infty$, $i = 1, \dots, 6$;

(B2) $\exists \mu \in S_{rca} \;\; \mu = A\mu$, $\mu(F) > 0$.

Proof. By Theorem 11.2 and the subsequent remark, we may assume in (A2) that $r_n^i(F) \to 0$ as $n \to \infty$ for some $i \leq 6$.

If (B2) holds then, obviously, (A1) fails. Suppose now that (A1) is not fulfilled. By Theorem 11.3, there exists $\mu \in S_{rba}$ with $\mu = A\mu$, $\mu(F) > 0$. Decompose the measure $\mu = \mu_1 + \mu_2$ into its purely finitely additive component μ_1 and countably additive component μ_2. Since $0 \leq \mu_1, \mu_2 \leq \mu$, regularity of μ implies that μ_1 and μ_2 are regular, i.e., $\mu_1 \in rba(X, \mathcal{B})$ and $\mu_2 \in rca(X, \mathcal{B})$. By Alexandrov's Theorem (see [3, Chapter III, Section 5, Theorem 13]), a regular finitely additive measure on a compact set is countably additive. Indeed,

$\mu_1(F) = 0$, whence $\mu(F) = \mu_2(F) > 0$. Since $\mu = A\mu$, we have $\mu_1 = A\mu_1$ and $\mu_2 = A\mu_2$ (see [5]). So $\mu_2 \in rca(X, \mathcal{B})$, $\mu_2 = A\mu_2$, and $\mu_2(F) > 0$, i.e., the normalized measure μ_2 satisfies (B2).

We now prove that (A1) is equivalent to (A2). Clearly, (A2) implies (A1). Assume that (A2) does not hold for $i = 6$. Then there exist $\delta > 0$, a sequence of measures $\{\mu_j\}$, $\mu_j \in S_{rba}$, $j = 1, 2, \dots$, and a strictly increasing sequence of indices $\{n_j\}$ such that

$$\lambda_{n_j}^{\mu_j}(F) \geq \delta, \quad j = 1, 2, \dots .$$

By Theorem 2.9, each τ_C-limit point μ of $\{\lambda_{n_j}^{\mu_j}\}$ meets the condition $\mu(F) \geq \delta$. Construct a sequence of measures $\{\eta_n\}$ as follows: $\eta_{n_j} = \mu_j$, $j = 1, 2, \dots$, and $\eta_n \in S_{rba}$ are arbitrary measures for $n \neq n_j$. Then $\lambda_{n_j}^{\mu_j} = \lambda_{n_j}^{\eta_{n_j}}$, i.e., $\{\lambda_{n_j}^{\mu_j}\}$ is a subsequence in $\{\lambda_n^{\eta_n}\}$, and hence all τ_C-limit measures μ for $\{\lambda_n^{\mu_j}\}$ are τ_C-limit for $\{\lambda_n^{\eta_n}\}$. By Theorem 7.4 (Corollary 7.2), all such μ are invariant for A. By Theorem 7.1, the set of all such measures is nonempty and included in S_{rba}. Thus if (A2) does not hold then there exists $\mu \in S_{rba}$, $\mu = A\mu$, $\mu(F) > 0$. Substituting μ for the initial measure in λ_n^{μ}, we see that (A1) is not fulfilled. The theorem is proven.

We emphasize that, *for a compact set F and a Feller* MC, *convergence $\lambda_n^{\mu}(F) \to 0$ as $n \to \infty$ for every $\mu \in S_{rba}$ implies uniform convergence in $\mu \in S_{rba}$ and $\mu \in S_{ba}$.*

The claims of Theorem 11.4 are partially contained in Foguel's articles [5, 6]. Therein, equivalence of (A1) and (A2) was proven for $i = 3$ and alternative (B2) was formulated for a compact Baire set. However, his proofs contain a gap: a τ_C-limit measure for a sequence of measures $\{\lambda_n\}$ is thought of as τ_C-limit for a subsequence λ_{n_i}. But this is not necessarily so even for a dense sequence of measures if no extra conditions are imposed on the compact sets (Prokhorov's Theorem (see [11]) does not hold for all spaces). Therefore, it should be proven that the τ_C-limit measures of the means of λ_n^{μ} (that, generally speaking, are not the limits of subsequences $\lambda_{n_i}^{\mu}$) are invariant. This fact was proven in Theorem 7.1 that was indirectly used in the proof of Theorem 11.4.

We now consider arbitrary MC's on arbitrary (X, Σ). The gamma-compactification γX of (X, Σ) is Hausdorff and compact and hence is normal. The isomorphic MC is Feller on $(\gamma X, \mathcal{B}_{\gamma X})$. This allows us to use Theorems 11.3 and 11.4 in the general case.

Theorem 11.5. *Suppose that we have an arbitrary* MC *on* (X, Σ). *Then, for every $E \in \Sigma$, Condition* (A3) *is equivalent to* (A4), *and we always have either* (A3), (A4) *or* (B3):

(A3) $\forall \mu \in S_{ba}\ \ \lambda_n^{\mu}(E) \to 0$ *as* $n \to \infty$;

(A4) $r_n^i(E) \to 0$ *as* $n \to \infty$, $i = 1, 2, 3, 4$;

(B3) $\exists \mu \in S_{ba}\ \ \mu = A\mu$, $\mu(E) > 0$.

Proof. By Theorem 11.1, we may assume in (A4) that $r_n^i(E) \to 0$ as $n \to \infty$ for some $i = 1, 2, 3, 4$.

Suppose that $E \in \Sigma$. Then $t(E) \in \mathcal{N}_{\gamma X}$ and $t(E)$ is compact in γX. Consequently, Theorem 11.4 holds for the γ-MC and the set $t(E)$. Moreover, since γX is compact, we have $rba(\gamma X, \mathcal{B}_{\gamma X}) = rca(\gamma X, \mathcal{B}_{\gamma X})$. From Section 10 it follows readily that Conditions (A3) and (B3) for an arbitrary $E \in \Sigma$ are equivalent to Conditions (A1) and (B2) for $F = t(E)$. We are left with proving equivalence of (A2) and (A4) for some $i \leq 4$.

Put $i = 1$ for the MC on (X, Σ) and $i = 5$ for the γ-MC on $(\gamma X, \mathcal{B}_{\gamma X})$ and denote the corresponding suprema by r_n^1 and $\widetilde{r}_n^5$.

Suppose that $\mu \in S_{ba}$ and $\widetilde{\mu} = [r^*]^{-1}\mu$. The constructions of Sections 9 and 10 imply that $\lambda_n^\mu(E) = \lambda_n^{\widetilde{\mu}}\big(t(E)\big)$, $\mu \in S_{ba}$, $E \in \Sigma$, where λ_n^μ corresponds to the MC and $\lambda_n^{\widetilde{\mu}}$, to the γ-MC. Since the sets S_{ba} in $ba(X, \Sigma)$ and S_{rca} in $rca(\gamma X, \mathcal{B}_{\gamma X})$ are isometrically isomorphic, we have the equality $r_n^1(E) = \widetilde{r}_n^5\big(t(E)\big)$. Consequently, $r_n^1(E) \to 0$ as $n \to \infty$ if and only if $\widetilde{r}_n^5\big(t(E)\big) \to 0$ as $n \to \infty$. The theorem is proven.

Note that, as for compact sets in the Feller case, *"pointwise" convergence* $\{\lambda_n^\mu(E)\}$ *(i.e., convergence for every* $\mu \in S_{rca}$*) implies uniform convergence in* $\mu \in S_{ba}$ *on every* $E \in \Sigma$.

For an arbitrary MC, we can now distinguish an analog of the *dissipative part* in X. Theorem 11.5 readily implies the following assertion.

Corollary 11.2. *Assume given an* MC *on* (X, Σ) *and* $K \in \Sigma$ *such that* $\mu(K) = 1$ *for all* $\mu \in \Delta$. *Then*

$$\sup_{\eta \in S_{ba}} \lambda_n^\eta(X \backslash K) = \sup_{\eta \in S_{ca}} \lambda_n^\eta(X \backslash K) = \sup_{x \in X} \frac{1}{n} \sum_{k=1}^{n} p^k(x, X \backslash K) \to 0$$

as $n \to \infty$.

In other words, the MC "disperses" on the "dissipative" set $X \backslash K$. It should be natural to try to maximize $X \backslash K$, to give the latter a more explicit form. In the next section we will see that this can be done if $\Delta \subset ca(X, \Sigma)$. Then, on $X \backslash K$, not only the means $\{\lambda_n^\mu\}$ vanish but also so does the very Markov sequence of measures $\{\mu_n\}$, and exponentially at that.

If the set of invariant measures Δ contains purely finitely additive measures then, as seen from Theorems 6.1 and 6.2 and Example 6.1, K need not have a subset stochastically closed with respect to the transition function. This neither allows us to distinguish K more explicitly nor provides stronger assertions about convergence on $X \backslash K$.

Theorem 11.5 can also be applied to a Feller MC. Combining it with Theorem 11.3, we infer that if $F \subset X$ is a closed subset meeting (A1) then the MC has no regular finitely additive invariant measure μ with $\mu(F) > 0$.

However, in this case (if (A4) does not hold), the MC can have a nonregular finitely additive measure η such that $\eta(F) > 0$.

With the alternative existence theorems available, it is natural to settle the following question. Suppose that the conditions of type (A) are not fulfilled. What connection exists between the limit behavior of the Cesàro means $\{\lambda_n^\mu\}$ and invariant measures nondegenerate on the corresponding sets? In essence, we have already got an answer to this question in the theorems of Sections 2 and 7. We expose these results in a suitable form below, and, after each formulation, we will point out explicitly where they are deduced from.

Theorem 11.6. *Let* (X, Σ) *and an* MC *be arbitrary. Then, for every* $\mu \in S_{ba}$*, there exists* $\eta \in \Delta_{ba}$ *such that, for every* $E \in \Sigma$*, there exists a subsequence* $\{n_i\} = \{n_i\}(E)$ *with* $\lambda_{n_i}^\mu(E) \to \eta(E)$ *as* $i \to \infty$.

If some $\mu \in S_{ba}$*, subsequence* $\{n_j\}$*,* $E \in \Sigma$*, and* $\alpha, \beta \in [0,1]$ *satisfy the inequalities* $\alpha \le \lambda_{n_j}^\mu(E) \le \beta$*,* $j = 1, 2, \ldots$*, then there exists* $\eta \in \Delta_{ba}$ *such that* $\alpha \le \eta(E) \le \beta$.

Taking into account Definition 2.2 and the definition of $\tau_{\mathcal{B}}$-topology in Section 1, Theorem 11.6 follows from Theorems 2.7 and 7.7.

Theorem 11.7. *Let* X *be normal. Assume given a Feller* MC *on* $(X, \mathcal{B})$. *Then, for every* $\mu \in S_{rba}$*, there exists* $\eta \in \Delta_{rba}$ *such that*

(1) $\forall f \in C(X) \quad \exists \{n_i\} = \{n_i\}(f) \quad \int f\, d\lambda_{n_i}^\mu \to \int f\, d\eta$;

(2) $\forall G = \overset{\circ}{G} \subset X,\ \eta(G) = \eta(\overline{G}),\ \ \exists \{n_i\} = \{n_i\}(G) \quad \lambda_{n_i}^\mu(G) \to \eta(G)$;

(3) $\forall F = \overline{F} \subset X,\ \ \eta(F) \ge \underline{\lim}\, \lambda_n^\mu(F);\ \ \forall G = \overset{\circ}{G} \subset X \quad \eta(G) \le \overline{\lim}\, \lambda_n^\mu(G)$.

Reckoning with the definition of τ_C-topology in Section 1, Theorem 11.7 follows from Theorem 7.1, Alexandrov's Theorem (see [3, Chapter IV, Section 9, Theorem 15]), and Theorem 2.9.

Theorem 11.8. *Suppose that* X *is a Hausdorff compact space and we have a Feller* MC *on* $(X, \mathcal{B})$. *Then, for every* $\mu \in S_{rca}$*, there exists* $\eta \in \Delta_{rca}$ *satisfying assertions* (1)–(3) *of Theorem* 11.7.

The proof is based on Alexandrov's Theorem stating that every regular finitely additive measure is countably additive on a compact space.

Theorem 11.9. *Suppose that* X *is a compact metric space and we have a Feller* MC *on* $(X, \mathcal{B})$. *Then, for every* $\mu \in S_{rca}$*, there exist* $\eta \in \Delta_{rca}$ *and* $\{n_i\}$ *such that* $\lambda_{n_i}^\mu \to \eta$ *in the* τ_C*-topology (i.e.,* $f(\lambda_{n_i}^\mu) \to f(\eta)$ *for all* $f \in C(X)$*).*

Theorem 11.9 follows from Theorem 7.1 and the well-known Prokhorov's Theorem (see [11]).

Recall (see [11]) that a family of measures $M \subset S_{rca}$ is called *dense* if, for every $\varepsilon > 0$, there exists a compact set $K \subset X$ with $\mu(K) \ge 1 - \varepsilon$ for all $\mu \in M$.

Theorem 11.10. *Suppose that X is a metric space, the* MC *is Feller, and the sequence of measures $\{\lambda_n^\mu\}$ is dense for some $\mu \in S_{rca}$. Then there exist $\eta \in \Delta_{rca}$ and $\{n_i\}$ such that $\lambda_{n_i}^\mu \to \eta$ in the τ_C-topology.*

Theorem 11.10 also follows from Theorem 7.1 and Prokhorov's Theorem (see [11]).

Let X be normal. Then, for every $\lambda \in ba(X, \mathcal{A})$, there exists a unique measure $\overline{\lambda} \in rba(X, \mathcal{A})$ such that $\int f d\lambda = \int f d\overline{\lambda}$ for every $f \in C(X)$ (see [19]). We call $\overline{\lambda}$ the *regularization of* λ. Suppose that $\mu \in rba(X, \mathcal{A})$ and $\mu \geq 0$. We call the set $\mathcal{R}(\mu) = \{\lambda \in ba(X, \mathcal{A}) : \lambda \geq 0, \overline{\lambda} = \mu\}$ the *class of C-equivalent measures for μ* (see [19] and [21, Section 1]).

Theorem 11.11. *Suppose that X is normal, the* MC *is arbitrary, $\mu \in S_{rca}$, and η is the τ_B-limit point for $\{\lambda_n^\mu\}$ (and hence $\eta \in \Delta_{ba}$). Then there exists $\zeta \in S_{rba}$ such that $\zeta = \bar{\eta}$ and ζ is a τ_C-limit point for $\{\lambda_n^\mu\}$. If, in addition, the* MC *is Feller then $\zeta = A\zeta$, i.e., $\zeta \in \Delta_{rba}$.*

Theorem 11.11 follows from Theorems 7.2, 2.7, 2.8, and 7.1.

Theorem 11.12. *Suppose that X is a metric space, the* MC *is arbitrary, $\mu \in S_{rca}$, and $\{\lambda_n^\mu\}$ is dense. Then, for every τ_B-limit (consequently, belonging to Δ_{ba}) measure η there exist $\{n_i\}$ and a base β of the initial topology of X such that $\lambda_{n_i}^\mu(E) \to \eta(E)$ for every $E \in \beta$. Moreover, the regularization $\bar{\eta} \in rca(X, \mathcal{B})$ is τ_C-limit for $\{\lambda_n^\mu\}$ and $\lambda_{n_i}^\mu \to \bar{\eta}$ in the τ_C-topology. If the* MC *is Feller then $\bar{\eta} = A\bar{\eta}$.*

Theorem 11.12 follows from Prokhorov's Theorem [11], Theorems 2.5, 2.7, and 2.8 [21], Theorem 6 [3], and Theorem 7.1 [21].

12. Strong Limit Theorems

Since the first articles by Doeblin in the beginning of the 20th century, convenient conditions have been sought for MC's to have rather good behavior: Markov sequences of measures or their means converge to combinations of invariant measures in a metric topology. The most general condition of the kind is Condition (D) by Doob and Doeblin (see [4]) which generalizes Doeblin's conditions from a countable phase space to an arbitrary phase space. Later Yosida and Kakutani (see [15]) and others proved that (D) is equivalent to quasicompactness Conditions (K1)–(K3) for the Markov operators. Recall that an operator T is called *quasicompact* (*quasicompletely continuous*) if there exist a compact (completely continuous) operator T_1 and an integer $k \geq 1$ such that $\|T^k - T_1\| < 1$.

Although exhaustive in a sense, the Doob–Doeblin Conditions are of a purely analytical nature, which complicates their use in practice. This encourages many investigators to look either for equivalent but simpler conditions or for conditions close to (D) but more convenient for specific problems.

We now formulate the above-mentioned equivalent conditions for an arbitrary MC on an arbitrary phase space (X, Σ):

(D) There exist $\varphi \in ca(X, \Sigma)$, $\varphi \geq 0$, $\varepsilon > 0$, and $k \geq 1$ such that $\varphi(E) \leq \varepsilon$, $E \in \Sigma$, implies $p^k(x, E) \leq 1 - \varepsilon$ for all $x \in X$;

(K1) $T\colon B(X, \Sigma) \to B(X, \Sigma)$ is quasicompact;

(K2) $A\colon ba(X, \Sigma) \to ba(X, \Sigma)$ is quasicompact;

(K3) $A\colon ca(X, \Sigma) \to ca(X, \Sigma)$ is quasicompact.

If the Markov operators of the chain are quasicompact then we call the MC itself *quasicompact.*

If, on (X, Σ), we have an MC with transition function $p(x, E)$ and Markov operators T and A then, for every $m \geq 1$, we can define a new MC with transition function $q_m(x, E)$ and operators T_m and A_m as follows:

$$q_m(x, E) = \frac{1}{m} \sum_{k=1}^{m} p^k(x, E), \quad T_m = \frac{1}{m} \sum_{k=1}^{m} T^k, \quad A_m = \frac{1}{m} \sum_{k=1}^{m} A^k.$$

We call these MC's *finitely averaged* MC*'s* (by the initial MC).

We now state the following condition $(\widetilde{\mathrm{D}})$ for the family of finitely averaged MC's:

$(\widetilde{\mathrm{D}})$ There exist $\varphi \in ca(X, \Sigma)$, $\varphi \geq 0$, $\varepsilon > 0$, and $m \geq 1$ such that from $\varphi(E) \leq \varepsilon$, $E \in \Sigma$, it follows that $q_m(x, E) \leq 1 - \varepsilon$ for all $x \in X$.

Clearly, $(\widetilde{\mathrm{D}})$ is Doob–Doeblin Condition (D) for the finitely averaged MC (with $m \geq 1$ fixed) with $k = 1$. Hence if $(\widetilde{\mathrm{D}})$ is fulfilled then T_m and A_m are quasicompact, i.e., they meet Conditions (K1)–(K3).

Theorem 12.1. *If an* MC *satisfies Doob–Doeblin Condition* (D) *then it also meets* $(\widetilde{\mathrm{D}})$.

Proof. Suppose that (D) holds for some φ, ε, and $k \geq 1$. We put $\tilde{\varepsilon} = \frac{\varepsilon}{k}$. Then

$$q_k(x, E) = \frac{1}{k} \sum_{i=1}^{k} p^i(x, E) \leq \frac{k-1}{k} + \frac{1}{k}(1 - \varepsilon) = 1 - \frac{\varepsilon}{k} = 1 - \tilde{\varepsilon}$$

if $\varphi(E) \leq \tilde{\varepsilon} \leq \varepsilon$. Consequently, Condition $(\widetilde{\mathrm{D}})$ holds for the same measure φ, $\tilde{\varepsilon} = \frac{\varepsilon}{k}$, and $m = k$. The theorem is proven.

We amplify our information about finitely averaged MC's with the following assertion.

Lemma 12.1. *If* $(\widetilde{\mathrm{D}})$ *holds for some* $m = m_0$ *then it also holds for all* $m \geq m_0$.

Proof. Suppose that $(\widetilde{\mathrm{D}})$ is satisfied for some φ, ε, and m. Repeating the calculations of the preceding theorem, we infer that

$$q_{m+1}(x,E) = \frac{m}{m+1} q_m(x,E) + \frac{1}{m+1} p^{m+1}(x,E).$$

Put $\tilde{\varepsilon} = \frac{m}{m+1}$. If $\varphi(E) \le \tilde{\varepsilon} \le \varepsilon$ then we have $q_{m+1}(x,E) \le 1-\tilde{\varepsilon}$.

Therefore, Condition $(\widetilde{\mathrm{D}})$ holds for $m+1$ with the same φ and $\tilde{\varepsilon} = \frac{m}{m+1}$. By induction, the lemma follows for all $m+k$. The lemma is proven.

Corollary 12.1. *If the initial* MC *is quasicompact then all its finitely averaged* MC*'s are quasicompact from some number m on.*

Note that $q_{m+1}(x,E)$ is not a convolution of $q_m(x,E)$ but generates an MC of its own.

Denote by $\widetilde{\Delta}_m$ the family of all normed positive finitely additive invariant measures for the finitely averaged MC with parameter m. Obviously, we have $\Delta = \Delta_{ba} \subset \widetilde{\Delta}_m$ for every $m \in \mathbb{N}$.

Theorem 12.2. *For every* MC, $(\widetilde{\mathrm{D}})$ *is equivalent to the following condition*:

$$\Delta_{ba} \subset ca(X,\Sigma), \qquad (*)$$

i.e., the finitely averaged MC *(for some $m \ge 1$) is quasicompact if and only if all invariant finitely additive measures of the initial chain are countably additive, or, in other words, if and only if the initial* MC *has no nonzero invariant purely finitely additive measures.*

Proof. Suppose that $(*)$ is fulfilled. Then $\dim \Delta = n < \infty$ by Theorem 8.2. Let $\{\mu_1,\dots,\mu_n\}$ be a singular basis for Δ, which exists by Theorem 6.3. Put $\varphi = \mu_1 + \dots + \mu_n$. Assume that φ does not satisfy $(\widetilde{\mathrm{D}})$. Then, for every $m \ge 1$ and $\varepsilon > 0$, there exist $E_{m,\varepsilon} \in \Sigma$ and $x_{m,\varepsilon} \in X$ such that $\varphi(E_{m,\varepsilon}) \le \varepsilon$ and $q_m(x_{m,\varepsilon}; E_{m,\varepsilon}) > 1-\varepsilon$. For every $m = 1,2,\dots$ and some fixed $\delta \in (0,1)$, put

$$\varepsilon = \varepsilon(m) = \frac{\delta}{2^m}, \quad E_\delta = \bigcup_{m=1}^{\infty} E_{m,\varepsilon(m)}.$$

Then

$$\begin{aligned} &\varphi(E_\delta) \le \sum \varphi\Big(E_{m,\varepsilon(m)}\Big) < \delta \sum \frac{1}{2^m} = \delta, \\ &q_m\Big(x_{m,\varepsilon(m)};\, E_\delta\Big) \ge q_m\Big(x_{m,\varepsilon(m)};\, E_{m,\varepsilon(m)}\Big) > 1 - \frac{\delta}{2^m}, \end{aligned} \qquad m = 1,2,\dots.$$

Take a sequence $\eta_m \in S_{ca}$ of Dirac measures $\eta_m = \delta_{x_{m,\varepsilon(m)}}$, $m = 1,2,\dots$. Then

$$\lambda_m^{\eta_m}(E_\delta) = \frac{1}{m}\sum_{s=1}^{m} A^s \eta_m(E_\delta) = A_m\eta_m(E_\delta) = q_m\Big(x_{m,\varepsilon(m)}; E_\delta\Big) > 1 - \frac{\delta}{2^m}$$

for $m = 1,2,\dots$.

By Theorem 7.3, the set $\mathfrak{M}\{\lambda_m^{\eta_m}\}$ of all τ_B-limit points of the sequence $\{\lambda_m^{\eta_m}\}$ is nonempty and included in the set of invariant measures Δ. Suppose that $\mu \in \mathfrak{M}\{\lambda_m^{\eta_m}\} \subset \Delta$. Recall that if the limit $\lim \lambda_m^{\eta_m}(E)$ exists then $\mu(E) = \lim \lambda_m^{\eta_m}(E)$. The above estimate implies that $\mu(E_\delta) = \lim \lambda_m^{\eta_m}(E_\delta) = 1$. So, if $(\widetilde{\mathrm{D}})$ is not fulfilled for $\varphi = \mu_1 + \dots + \mu_n$ then there exist $\delta \in (0,1)$, $E_\delta \in \Sigma$, and $\mu \in \Delta_{ba}$ such that $\varphi(E_\delta) < \delta < 1$ and $\mu(E_\delta) = 1$.

Since $\{\mu_1, \dots, \mu_n\}$ is an orthonormal basis for Δ and $\mu \in \Delta$, we have $\mu = \sum_{i=1}^n \alpha_i \mu_i$, where $0 \le \alpha_i \le 1$ for $i = 1, 2, \dots, n$. Hence $\mu \le \sum_{i=1}^n \mu_i = \varphi$. Consequently, $\mu(E_\delta) \le \varphi(E_\delta) < \delta < 1$. The contradiction proves that $(*)$ implies fulfillment of $(\widetilde{\mathrm{D}})$ with $\varphi = \mu_1 + \dots + \mu_n$.

Suppose that $(\widetilde{\mathrm{D}})$ holds for some $\varphi \in ca(X, \Sigma)$, $\varepsilon > 0$, and $m \ge 1$. Assume that $(*)$ does not hold. Then the MC has an invariant purely finitely additive measure $\lambda \in S_{ba}$. Every purely finitely additive measure is disjoint from every countably additive measure (see [16]). Moreover, for each $\varepsilon > 0$, there exists a set $G \in \Sigma$ such that $\varphi(G) \le \varepsilon$ and $\lambda(X \backslash G) \le \frac{\varepsilon}{2}$ (see [16, Theorem 1.21]). Now, reckoning with Theorem 11.1, we infer

$$\sup_{x \in X} q_m(x, G) = \sup_{\mu \in S_{ba}} A_m \mu(G) \ge A_m \lambda(G) = \lambda(G) \ge 1 - \varepsilon/2 > 1 - \varepsilon,$$

which contradicts $(\widetilde{\mathrm{D}})$ for the chosen $m \ge 1$. Therefore, $(*)$ holds. The theorem is proven.

Corollary 12.2. *Suppose that an* MC *satisfies* (D). *Then it also meets* $(*)$.

Theorem 8.3 and the proof of Theorem 12.2 immediately imply the following assertion.

Theorem 12.3. *Suppose that* $\dim \Delta_{ba} = 1$ *for an arbitrary* MC. *Then* $(\widetilde{\mathrm{D}})$ *holds, i.e., the finitely averaged* MC*'s are quasicompact from some* m *and* $\Delta_{ba} = \Delta_{ca} = \{\mu\} \subset ca(X, \Sigma)$.

In Revuz's monograph, there is a result (see [13, Chapter 6, Section 3, Theorem 3.10]) going back to Horowitz (see [8]) in which the limit behavior of the MC is also connected with invariant finitely additive measures. However, therein, stringent conditions are a priori imposed on the MC: a Harris chain is defined with a prescribed invariant countably additive measure on a separable $(X, \mathcal{B})$. It is proven that, in this case, quasicompactness of the MC is equivalent to absence of invariant purely finitely additive measures (moreover, the above invariant measure is unique). As we can see, in Theorems 12.2 and 12.3 close in content none of these conditions is supposed to hold.

The asymptotic behavior of quasicompact MC's, i.e., of MC's meeting the Doob–Doeblin Condition has been completely studied. Therefore, we will not write out numerous and cumbersome corollaries for the corresponding MC's. If need be, this is easy by rewriting, for example, the relevant parts of [4, 10]. We stress the key point: the conditions of Theorems 12.2 and 12.3 guarantee existence of strong limits in the space of measures for the corresponding MC's.

Thus, we have necessary and sufficient conditions for quasicompactness of an MC *in analytical form.* These are Doob–Doeblin Conditions (D) and Condition $(*)$ of quasicompactness of the finitely averaged MC, which we call *structural* (not in the sense of the terminology of structure or lattice theory but in the broad sense of the word "structure"). At the same time, Theorems 6.6 and 7.8 of the first part and Theorem 12.2 make it possible to formulate new analytical necessary and sufficient conditions for quasicompactness of an MC.

Theorem 12.4. *Assume given an* MC *on an arbitrary measure space* (X, Σ). *For the finitely averaged* MC *not to be quasicompact* (*for all* m), *it is necessary and sufficient that the following conditions hold for all* $m \in \mathbb{N}$:

$$\text{There exist } \varepsilon_n \geq 0, \ \ \varepsilon_n \to 0 \ \text{ as } \ n \to \infty,$$

$$\text{and } K_n \in \Sigma, \ \ K_n \neq \varnothing \ \text{ for } \ n \in \mathbb{N}, \ K_1 \supset K_2 \supset \cdots, \ \bigcap_{n=1}^{\infty} K_n = \varnothing, \qquad (\neg *)$$

$$\text{such that } \ q_m(x, K_n) \geq 1 - \varepsilon_n \ \text{ for } \ x \in K_{n+1}, \ n \in \mathbb{N}.$$

Theorem 12.4 is in fact a direct corollary to the above-mentioned theorems. Theorem 6.6 provides necessity of $(\neg *)$ for existence of an invariant purely finitely additive measure, Theorem 7.8 gives sufficiency, and Theorem 12.2 states equivalence of quasicompactness of the finitely averaged MC to absence of nonzero invariant purely finitely additive measures for the initial MC.

Corollary 12.3. *If an* MC *meets* $(\neg *)$ *then it is not quasicompact.*

We point out some technical aspects in $(\neg *)$, $(\widetilde{\mathrm{D}})$, and (D). As for the final result, for example, for a finitely averaged chain, it does not matter which of Conditions $(\widetilde{\mathrm{D}})$ and $(\neg *)$ to check for a specific MC because both conditions are necessary and sufficient and complementary to one another. Condition $(\neg *)$ excludes fulfillment of (D) for the initial chain. At the same time, for practically checking fulfillment of (D) or $(\widetilde{\mathrm{D}})$, it is necessary to find a suitable measure $\varphi \in ca(X, \Sigma)$ which causes problems. The matter is that φ is actually a linear combination of countably additive measures invariant for the MC, and they are such measures that are to be found.

If we look at $(\neg *)$ closely then we see that it contains no "extraneous" measures and only the transition function itself is involved. We believe that, in some situations, these features of $(\neg *)$ can facilitate the study of specific MC's.

13. Weak Limit Theorems

Convergence of an MC in the τ_C-topology, i.e. weak convergence in the probabilistic terminology, is closely connected with invariant purely finitely additive measures. In this section, we expose several results concerning this matter. The following theorem is of the same importance for weak convergence of an MC as Theorem 12.4 for strong convergence. The main peculiarity of Theorem 13.1 is that we do not presuppose existence of an invariant countably additive (i.e., classical "probability") measure for the MC. We proved a special case of Theorem 13.1 in [17]. In the form presented below, Theorem 13.1 was first published in the preprint [20].

Theorem 13.1. *Suppose that X is a normal topological space, we have an arbitrary* MC *on $(X, \mathcal{B})$, and $\mu \in S_{rca}$ is a countably additive ("probability") measure. For "weak" convergence of the sequence of the means $\{\lambda_n^\eta\}$ to μ, i.e., convergence in the τ_C-topology, for every initial finitely additive measure $\eta \in S_{ba}$, it is necessary and sufficient that*

$$\int f\, d\zeta = \int f\, d\mu \quad \text{for all} \quad \zeta \in \Delta \quad \text{and } f \in C(X), \tag{$**$}$$

or, in other words, $\Delta \subset \mathcal{R}(\mu)$, i.e., all invariant measures have μ as regularization.

Proof. The necessity is obvious. Indeed, let there exist $\zeta \in \Delta$ with $\bar{\zeta} \neq \mu$. Then, taking ζ as the initial measure, we have $\lambda_n^\zeta \equiv \zeta \to \zeta$ and $\bar{\zeta} \neq \mu$, i.e., λ_n^ζ does not converge to μ in the τ_C-topology.

We now prove the sufficiency. Let Condition $(**)$ hold for some $\mu \in S_{rca}$. Assume that $\lambda_n^\eta \nrightarrow \mu$ in the τ_C-topology for some $\eta \in S_{ba}$. Then, by Alexandrov's Theorem (see [3, Chapter IV, Section 9, Theorem 15]), there exists a set $G = \overset{\circ}{G}$ such that $\mu(G) = \mu(\overline{G})$ and $\lambda_n^\eta(G) \nrightarrow \mu(G)$, i.e., there exist $\varepsilon > 0$ and a strictly increasing sequence $\{n_i\}$ such that $\lambda_{n_i}^\eta(G) \geq \mu(G) + \varepsilon$ $\big($or $\leq \mu(G) - \varepsilon\big)$ for $i = 1, 2, \ldots$. Let ζ be a τ_B-limit point of $\lambda_{n_i}^\eta$. It exists by Theorem 7.2 and Corollary 7.2. Then $\zeta(G) \geq \mu(G) + \varepsilon$ and, moreover, $\zeta \in \Delta$ by Theorem 7.2.

Since $\bar{\zeta}$ is regular, for every $\delta > 0$, we can find a set $F = \overline{F} \subset G$ such that $\zeta(F) \geq \zeta(G) - \delta$. The difference $X \backslash G$ is closed and $(X \backslash G) \cap F = \varnothing$; therefore, by the Urysohn theorem (see [3, Chapter I, Section 5, Theorem 2]), there exists a function $f \in C(X)$, $0 \leq f(x) \leq 1$, with $f(F) = 1$ and $f(X \backslash G) = 0$.

Estimate the following integrals:

$$\begin{aligned} \int_X f d\zeta \geq \int_F f d\zeta \geq \zeta(F) \geq \zeta(G) - \delta \geq \mu(G) + \varepsilon - \delta \\ \geq \int_G f d\mu + \varepsilon - \delta = \int_X f d\mu + \varepsilon - \delta. \end{aligned}$$

Put $\delta = \frac{\varepsilon}{2}$. Then $\int_X f d\zeta \geq \int_X f d\mu + \frac{\varepsilon}{2}$, i.e., $\int f d\bar{\zeta} = \int f d\zeta \neq \int f d\mu$ and $\bar{\zeta} \neq \mu$, which contradicts $(**)$.

Consider the other possible case, $\lambda^{\eta}_{n_i}(G) \leq \mu(G) - \varepsilon$ for $i = 1, 2, \dots$. Then a τ_B-limit point ξ of $\lambda^{\eta}_{n_i}$ satisfies $\xi(G) \leq \mu(G) - \varepsilon$ and $\xi \in \Delta$. Since μ is regular, for every $\delta > 0$, there exists a set $F = \overline{F} \subset G$ such that $\mu(F) \geq \mu(G) - \delta$, i.e., $\mu(G) \leq \mu(F) + \delta$.

Take again a function $f \in C(X)$, $0 \leq f(X) \leq 1$, such that $f(F) = 1$ and $f(X \backslash G) = 0$.

We have

$$\int_X f d\xi \leq \int_G f d\xi \leq \xi(G) \leq \mu(G) - \varepsilon \leq \mu(F) - \varepsilon + \delta$$
$$= \int_F f d\mu - \varepsilon + \delta \leq \int_X f d\mu - \varepsilon + \delta.$$

Put $\delta = \frac{\varepsilon}{2}$ and obtain $\bar{\xi} \neq \mu$, which contradicts $(**)$. Consequently, in both cases, $\lambda^{\eta}_n \to \mu$ in the τ_C-topology for every $\eta \in S_{ba}$. The theorem is proven.

Corollary 13.1. *Under the conditions of* Theorem 13.1, *for the sequence* $\{\lambda^{\eta}_n\}$ *to converge weakly to a countably additive* (*"probability"*) *measure* $\mu \in S_{rca}$ *for every initial countably additive* (*"probability"*) *measure* $\eta \in S_{rca}$, *it is sufficient that* $(**)$ *hold.*

Condition $(**)$ is not necessary for the convergence $\lambda^{\eta}_n \to \mu$ in the τ_C-topology for every initial countably additive measure $\eta \in S_{rca}$ even if X is compact.

Example 13.1. Suppose that $X = [0, 1]$ and the MC is defined by the mapping $F \colon X \to X$,

$$F(x) = \begin{cases} x^2 & \text{for } x \in [0, 1), \\ 0 & \text{for } x = 1, \end{cases}$$

i.e., $p(x, E) = \delta_{x^2}(E)$ for $x \in [0, 1)$ and $p(1, E) = \delta_0(E)$.

Then $\lambda^{\eta}_n \to \delta_0$ in the τ_C-topology for every $\eta \in S_{rca}$. However, making use of theorems of Section 11, it is easy to prove that there exists a measure $\zeta \in \Delta$ such that $\zeta\big((1 - \varepsilon, 1)\big) = 1$ for all $\varepsilon > 0$, i.e., $\bar{\zeta} \neq \delta_0$.

If the MC is Feller then $(**)$ implies $\mu \in \Delta_{rca}$, i.e., μ is an invariant measure.

Theorem 13.2. *Let* X *be a Hausdorff compact space and let* $ca(X, \mathcal{B}) = rca(X, \mathcal{B})$. *Assume given a Feller* MC *on* $(X, \mathcal{B})$. *Then the following three conditions are equivalent*:

(1°) $\dim \Delta_{ca} = 1$, *i.e., the* MC *has a unique invariant countably additive measure* $\mu \in S_{rca}$;

(2°) *there exists* $\mu \in S_{rca}$ *such that* $\lambda_n^\eta \to \mu$ *in the* τ_C*-topology for all* $\eta \in S_{rca}$;

(3°) *there exists* $\mu \in S_{rca}$ *meeting* $(**)$, *i.e.*, $\Delta \subset \mathcal{R}(\mu)$, *or, in other words*, $\bar{\xi} = \mu$ *for all* $\xi \in \Delta$.

Proof. Let (2°) be satisfied. Clearly, the measure $\mu \in S_{rca}$ of (2°) is unique. By Theorem 13.1, μ meets $(**)$, i.e., it meets (3°).

By Theorem 13.1, fulfillment of (3°) with a measure μ implies fulfillment of (2°) with μ. So (2°) and (3°) are equivalent.

Assume that (3°) holds with some measure $\mu \in S_{rca}$, which is obviously unique. Take $\xi \in \Delta$ ($\Delta = \Delta_{ba} \neq \varnothing$ by Theorem 4.1). Since X is compact, $\bar{\xi}$ is countably additive. By hypothesis, the MC is Feller and hence $\bar{\xi} \in \Delta_{ca}$, i.e., $\bar{\xi} = \mu \in \Delta_{rca}$, and $\Delta_{ca} = \{\mu\}$. Thus (1°) holds.

Suppose now that (1°) holds with an invariant measure $\mu \in S_{rca}$. Assume that (3°) is not fulfilled. Then, for the measure μ of (1°), there exists $\xi \in \Delta$ such that $\bar{\xi} \neq \mu$. Since X is compact, it follows that $\bar{\xi}$ is countably additive and hence $\bar{\xi} \in \Delta_{ca}$. Then $\dim \Delta_{ca} \geq 2$, which contradicts (1°).

Thus (1°) and (3°) are also equivalent. The theorem is proven.

Recall that a Feller MC defined on a Hausdorff compact space always has an invariant countably additive ("probability") measure.

If, under the conditions of Theorem 13.1, Δ does not contain purely finitely additive measures then Condition $(*)$ of Theorem 12.2, equivalent to $(\widetilde{\mathrm{D}})$, holds.

Corollary 13.2. *Let the conditions of Theorem* 13.1 *be satisfied. If we have* $\lambda_n^\eta \to \mu$ *in the* τ_C*-topology for every* $\eta \in S_{rca}$ *and* λ_n^η *does not converge to* μ *in the* τ_B-, τ_{ba^*}-, *or* τ_{ba}*-topology at least for one* $\eta \in S_{ba}$ *then the* MC *has an invariant purely finitely additive measure.*

Figuratively, invariant purely finitely additive measures are a "buffer" near the limit countably additive measure μ (possibly invariant, and possibly being an "ejection" point for the operator A). If there is no "buffer" then the finitely averaged MC converges strongly to μ. In the presence of a "buffer," the MC converges weakly "sticking" in invariant purely finitely additive measures "stuck" to the limit measure μ in the τ_C-topology.

Remark. In [12], Ramakrishnan proved a finitely additive analog to Birkhoff's ergodic theorem. He considered a point transformation of the measure space preserving the finitely additive measure. Invariant finitely additive measures for point transformations were also studied by Chersi in [2] as well as by other authors. These investigations are close in spirit to this research but have no particular points of intersection with the results exposed here.

It is well known also that to study the usual determinate iterative processes generated by point transformations is convenient on describing them in the Markov chain language. In [18], we solve some specific problems in this connection in the light of the approach of this article.

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